# Kinematics & Dynamics of Linkages

## Lecture 5: Gears Kinematics

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Spring 2018

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# Fundamental Law of Gearing

The angular velocity ratio $m_v$ (gear ratio) is defined as the ratio of the output rotational speed divided by the input rotational speed:

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \pm \frac{r_{in}}{r_{out}} = \pm \frac{d_{in}}{d_{out}}$$


External set

Internal set

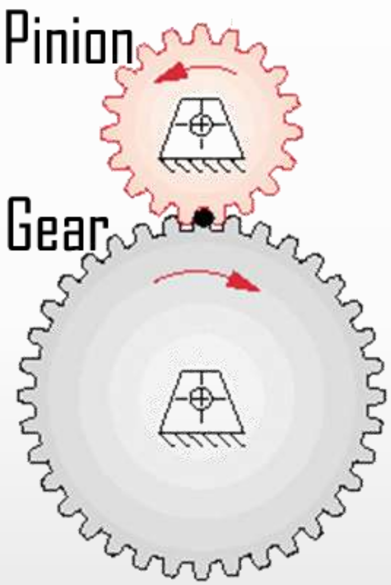
# Fundamental Law of Gearing

- Reduce velocity → increase torque (lower gears)
- Increase velocity → decrease torque (higher gears)

- External set of gears → reverses direction of rotation
- Internal sets of gears → rotate in same direction

# Fundamental Law of Gearing

- Force at A on the pinion $F_{A_{in}} = T_{in} / r_{in}$
- Force at A on the gear $F_{A_{out}} = T_{out} / r_{out}$
- Constraint $F_{A_{in}} = F_{A_{out}} \rightarrow r_{in} T_{in} = r_{out} T_{out}$
- Power in pinion $P_p = T_{in} \omega_{in}$
- Power in gear $P_g = T_{out} \omega_{out}$
- Power flow $T_{in} \omega_{in} = T_{out} \omega_{out}$



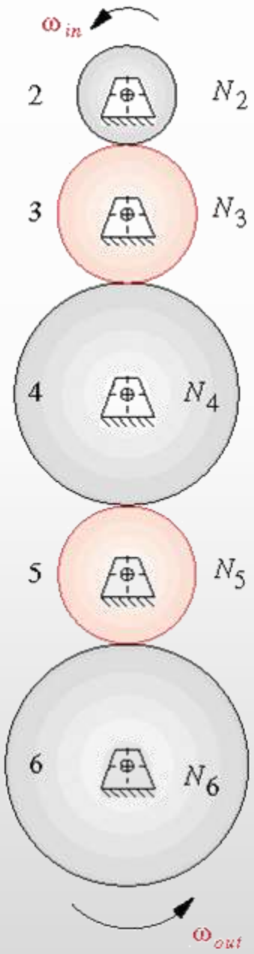

- Torque ratio $m_T = \dfrac{T_{in}}{T_{out}} = \dfrac{\omega_{out}}{\omega_{in}} = \pm \dfrac{r_{in}}{r_{out}} = \pm \dfrac{d_{in}}{d_{out}}$

# Simple Gear Trains

- Collection of 2 or more meshing gears
- **Simple gear train** is when there is 1 gear per shaft

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_3}{N_4}\right)\left(-\frac{N_4}{N_5}\right)\left(-\frac{N_5}{N_6}\right) = \left(+\frac{N_2}{N_6}\right)$$

- **External gears**
  - Odd number $\longrightarrow$ output same direction as input
  - Even number $\longrightarrow$ output opposite to input
  - The numerical effects of all internal cancel out
  - Intermediate gears are called idlers



# Simple Gear Trains

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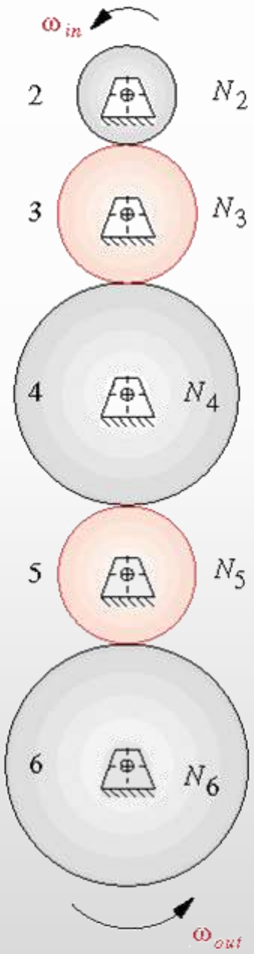
- **Limitation:** Single gear sets of spur, helical or bevel gears are usually limited to a velocity ratio of 10:1 simply because the gear sets would become very large



# Simple Gear Trains

## External gears

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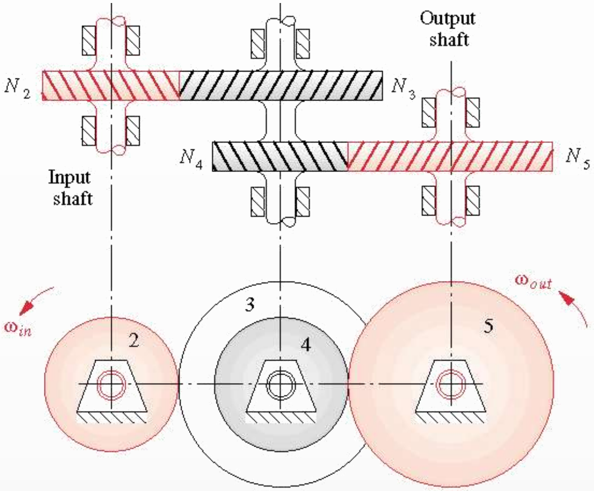
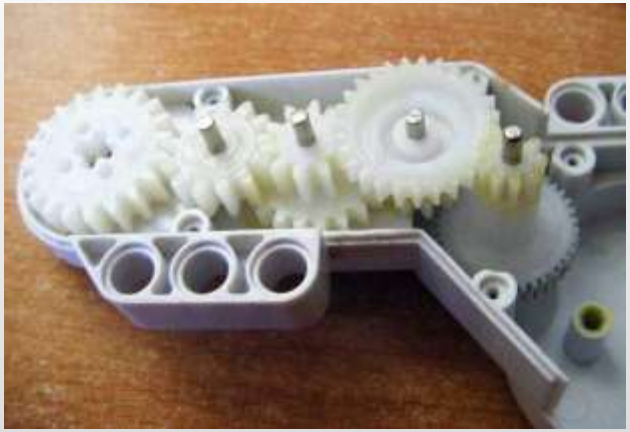
# Compound Gear Trains

- At least one shaft carries more than one gear
- Parallel or series–parallel arrangement
- Velocity ratios larger than 10:1 possible




$$m_v = \frac{\text{Product of \# of teeth on driver gears}}{\text{Product of \# of teeth on driven gears}}$$



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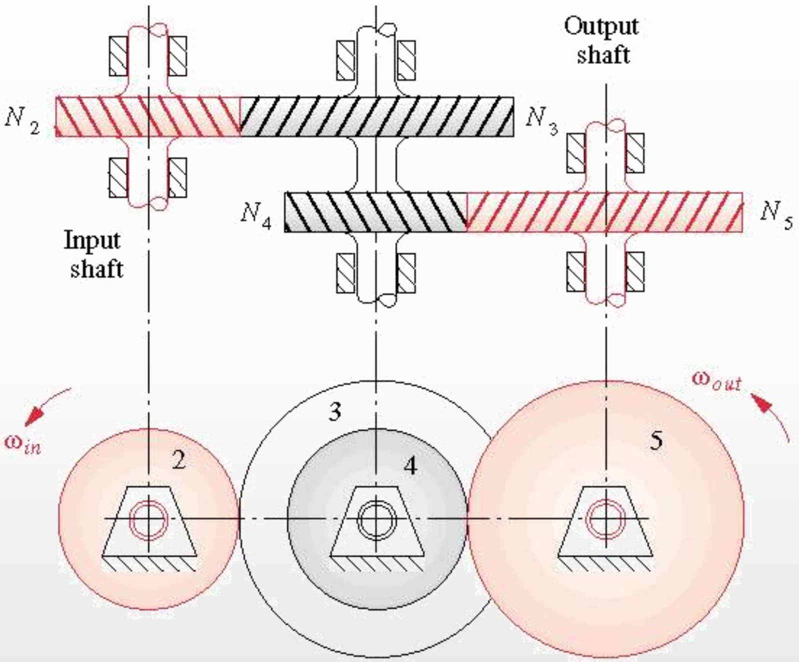
# Compound Gear Trains

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right)$$




Why?

$$\frac{\omega_3}{\omega_2} = \left(-\frac{N_2}{N_3}\right) \qquad \omega_2 = \left(-\frac{N_3}{N_2}\right)\omega_3$$

$$\frac{\omega_5}{\omega_4} = \left(-\frac{N_4}{N_5}\right) \qquad \omega_5 = \left(-\frac{N_4}{N_5}\right)\omega_4$$

$\omega_3 = \omega_4$ on the same shaft

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \frac{\omega_5}{\omega_2} = \frac{-\dfrac{N_2}{N_3}}{-\dfrac{N_5}{N_4}} = \frac{N_2}{N_3}\frac{N_4}{N_5}$$

# Design of Compound Gear Trains

Design a compound gear train for an exact velocity ratio of 180:1.
Find a combination of gears which will give that ratio.

**Step 1a:** Determine how many gear sets are needed

If 2 gear identical sets were selected, the individual gear ratio is

$$m_v = \frac{\omega_{out}}{\omega_{in}} = m_{v1} \times m_{v2} = 180 \Rightarrow m_{v1} = m_{v2} = \sqrt{180} = 13.416$$

Which is **not acceptable** since it is greater than 10

# Design of Compound Gear Trains

**Step 1b:**

If 3 gear identical sets were selected:

$$m_v = \frac{\omega_{out}}{\omega_{in}} = m_{v1} \times m_{v2} \times m_{v3} = 180 \Rightarrow m_{vi} = \sqrt[3]{180} = 5.646$$

Which is acceptable since it is less than 10

# Design of Compound Gear Trains

**Step 2:** Find gears that would satisfy the required ratio

The individual gear ratio should be 5.656 so

$$\frac{N_2}{N_3} = \frac{1}{5.646} \quad \text{or} \quad N_3 = 5.646 N_2$$

From the tables of 25° pressure angle gears and to avoid interference, a minimum of 12 teeth for the pinion is needed

# Design of Compound Gear Trains

**Step 2 (continued):**

Trying different integer teeth values:

$N_2$ = 14 teeth ⟶ $N_3$ = 79.05 ⟶ The closest to an integer (but not)

$N_2$ = 15 teeth ⟶ $N_3$ = 84.69

$N_2$ = 16 teeth ⟶ $N_3$ = 90.33

With 79:14 gear ratio, the output of the gear train is:

$$\frac{\omega_{in}}{\omega_{out}} = \left(-\frac{79}{14}\right)\left(-\frac{79}{14}\right)\left(-\frac{79}{14}\right) = -179.68 \quad \text{not exactly } 180$$

# Design of Compound Gear Trains

**Step 3:**

- Find the integer factors of 180 → 2, 2, 3, 3, 5
- Balance the sets as equally as possible while keeping ratios ≤ 10
- Since we know from previous steps that we need 3 sets with a ratio between 5 and 6, take multipliers close to this ratio (example: 5-6-6)

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right)\left(-\frac{N_6}{N_7}\right) = \left(-\frac{1}{5}\right)\left(-\frac{1}{6}\right)\left(-\frac{1}{6}\right) = -\frac{1}{180}$$

# Design of Compound Gear Trains

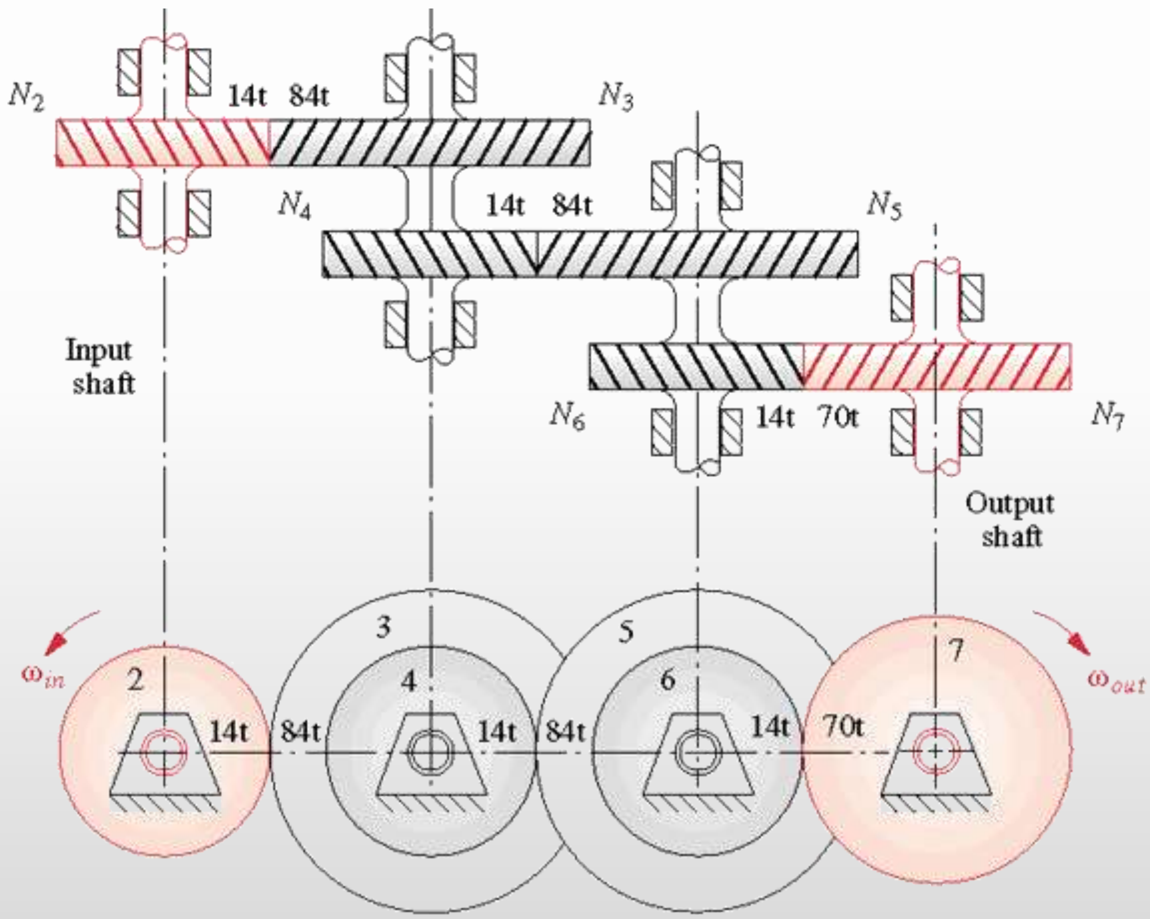
**Step 3 (continued):**

From the tables, using 3 pinions with 14 teeth is acceptable:

- $N_2 = N_4 = N_6 = 14$ teeth
- Then $N_3 = 70$, $N_5 = 84$, $N_7 = 84$ teeth

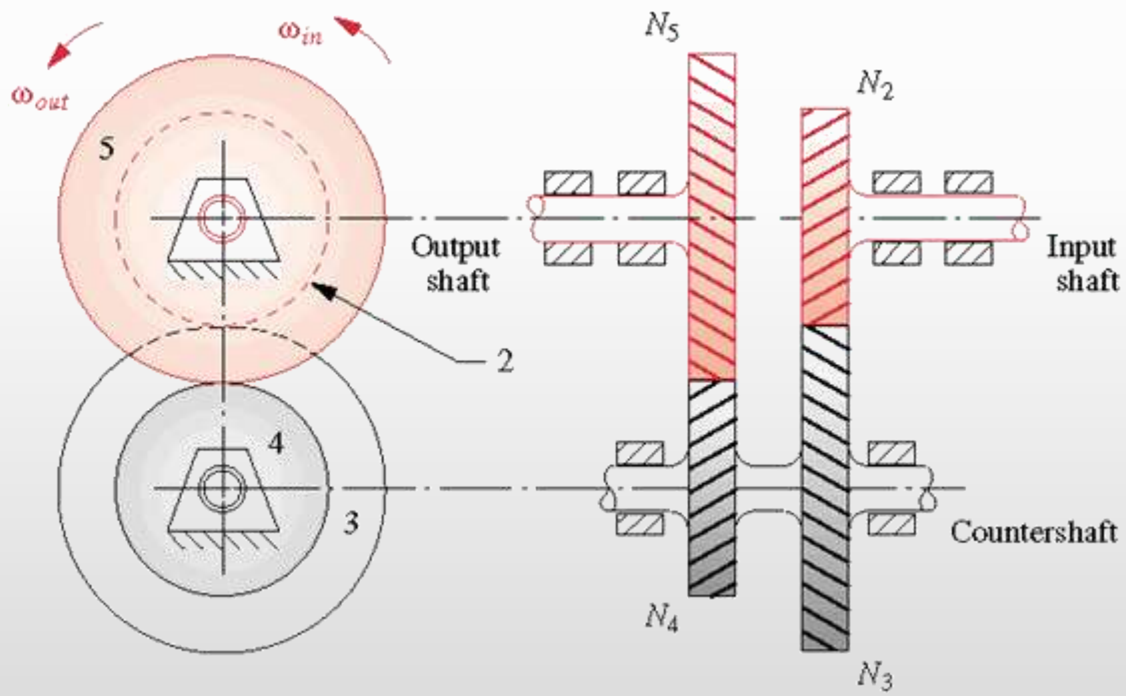
$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right)\left(-\frac{N_6}{N_7}\right) = \left(-\frac{14}{70}\right)\left(-\frac{14}{84}\right)\left(-\frac{14}{84}\right) = -\frac{1}{180}$$

# Design of Compound Gear Trains



# Reverted Gear Trains

- Compounded gear trains with concentric input/output shafts



# Reverted Gear Trains

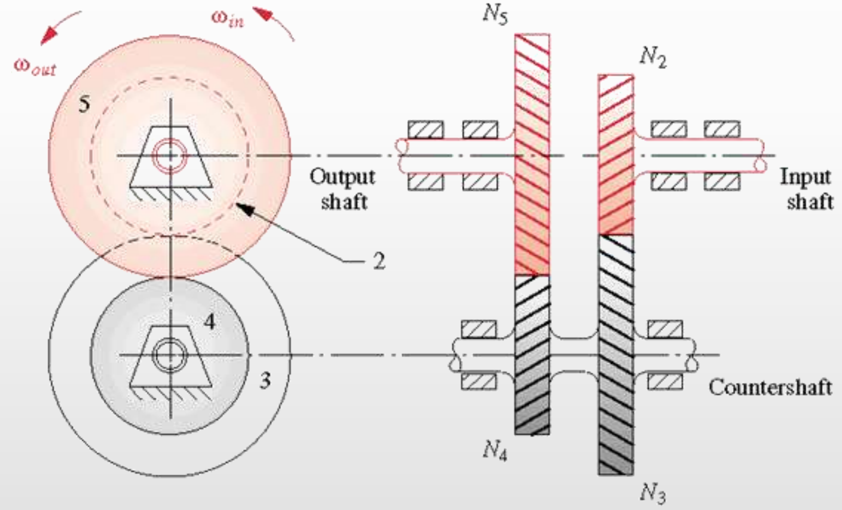
- To achieve a reverted gear train, the center distance between the gears should be equal thus the following constraint should be maintained:

$$r_2 + r_3 = r_4 + r_5$$

$$d_2 + d_3 = d_4 + d_5$$

$$but\ \rho_d = \frac{N}{d} \Rightarrow$$

$$N_2 + N_3 = N_4 + N_5$$



# Design of Reverted Gear Trains

Design a reverted compound gear train with an train ratio of 18:1

**Solution:** Start with two identical sets

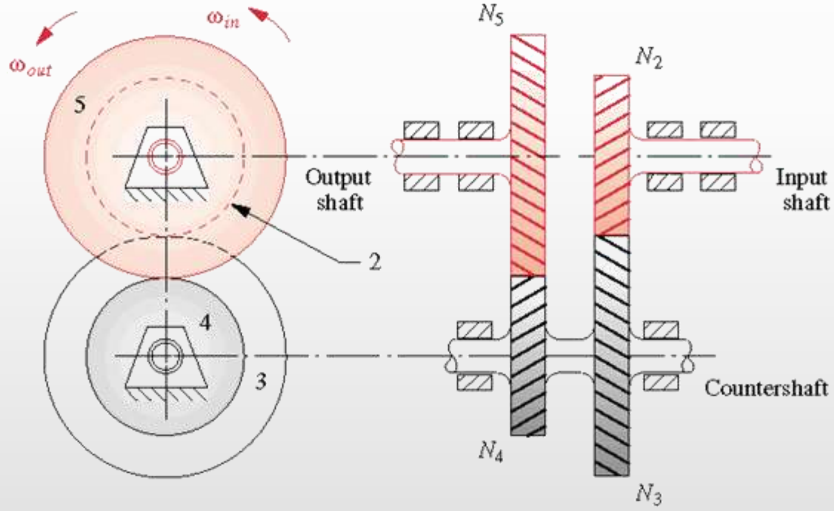
$$m_v = \frac{\omega_{out}}{\omega_{in}} = m_{v1} \times m_{v2} = 18 \Rightarrow m_{vi} \sqrt{18} = 4.24$$

Acceptable (<10) but not rational so no exact solution available

# Design of Reverted Gear Trains

If the pinions have 14 teeth, then:

- $N_2$ = 14 teeth $\longrightarrow$ $N_3$ = 59.40
- $N_2$ = 15 teeth $\longrightarrow$ $N_3$ = 63.64
- $N_2$ = 16 teeth $\longrightarrow$ $N_3$ = 67.88
- $N_2$ = 17 teeth $\longrightarrow$ $N_3$ = 72.12




No exact solution exists

# Design of Reverted Gear Trains

Instead, use the factors of 18 = 2 x 3 x 3

For 2 sets, the possible combinations are: 9 x 2 or 6 x 3

The best matched set would be 6 x 3, so:

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right) = \left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right) = \frac{1}{18}$$

# Design of Reverted Gear Trains

- Enforcing the reverted gear train constraint: $N_2 + N_3 = N_4 + N_5 = k$
- And the result that we got: $m_v = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right) = \left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right) = \frac{1}{8}$
- Then: $\frac{N_2}{N_3} = \frac{1}{6} \Rightarrow N_3 = 6N_2$

  $\frac{N_4}{N_5} = \frac{1}{6} \Rightarrow N_5 = 6N_4$

# Design of Reverted Gear Trains

- Substitute in gear train constraint and solving for k:

$$N_2 + N_3 = k \Rightarrow N_2 + 6N_2 = k \Rightarrow k = 7N_2$$
$$N_4 + N_5 = k \Rightarrow N_4 + 3N_4 = k \Rightarrow k = 4N_4$$

- Setting k to the lowest common multiple of 7 and 4 will give a valid solution for the equalities which is **28**

# Design of Reverted Gear Trains

- Using k=28 will result in a very small pinion (4 teeth)
- Increasing value of k by 2 (k=56) is still not enough (8 teeth)
- Increasing value of k by 4 (k=112) gives satisfactory results:

$$k = 7N_2 = 112 \Rightarrow N_2 = 16$$

$$k = 4N_4 = 112 \Rightarrow N_4 = 28$$

$$and$$

$$N_3 = 6N_2 = 96$$

$$N_5 = 3N_2 = 84$$

# Design of Reverted Gear Trains

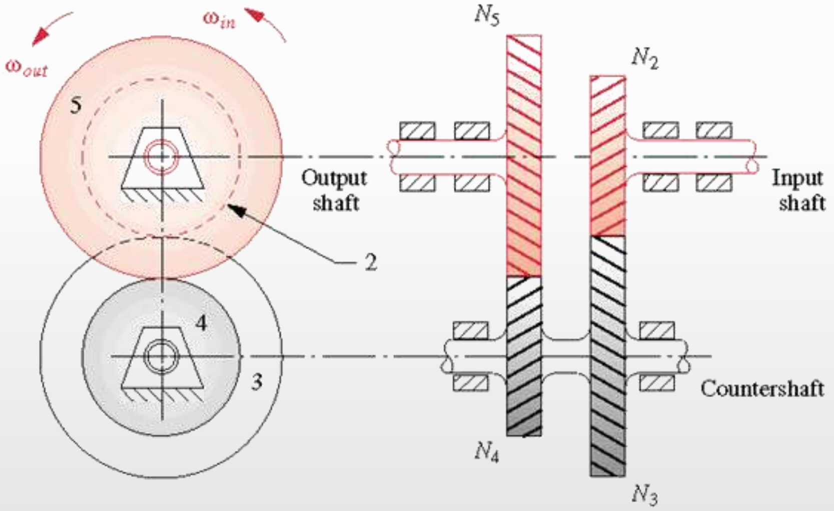
- So the final solution to the problem will be to use the following compounded gear sets:

$$N_2 = 16$$

$$N_3 = 96$$

$$N_5 = 84$$

$$N_4 = 28$$




- Which will result in

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right) = \left(-\frac{16}{96}\right)\left(-\frac{28}{84}\right) = \frac{1}{18}$$

# Design problem 2

Design a reverted compound gear train with an exact train ratio of 40:1 and a diametral pitch of 8.

**Solution:** Start with two identical sets

$$m_v = \frac{\omega_{out}}{\omega_{in}} = m_{v1} \times m_{v2} = 18 \Rightarrow m_{vi}\sqrt{40} = 6.32$$

Acceptable (<10) but not rational so no exact solution available

# Design problem 2

Instead, use the factors of 40 = 5 x 2 x 2 x 2

For 2 sets, the possible combinations are: 5x8 or 10x4

The best matched set would be 5x8, so:

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right) = \left(-\frac{1}{5}\right)\left(-\frac{1}{8}\right) = \frac{1}{40}$$

# Design problem 2

- Enforcing the reverted gear train constraint: $N_2 + N_3 = N_4 + N_5 = k$

- And the result that we got: $m_v = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right) = \left(-\frac{1}{5}\right)\left(-\frac{1}{8}\right) = \frac{1}{40}$

- Then: $\frac{N_2}{N_3} = \frac{1}{5} \Rightarrow N_3 = 5N_2$

$$\frac{N_4}{N_5} = \frac{1}{8} \Rightarrow N_5 = 8N_4$$

# Design problem 2

- Substitute in gear train constraint and solving for k:

$$N_2 + N_3 = k \Rightarrow N_2 + 5N_2 = k \Rightarrow k = 6N_2$$
$$N_4 + N_5 = k \Rightarrow N_4 + 8N_4 = k \Rightarrow k = 9N_4$$

- Setting k to the lowest common multiple of 6 and 9 will give a valid solution for the equalities which is **54**

# Design problem 2

- Using k=54 will result in a very small pinion (6 teeth)
- Increasing value of k by 2 (k=108) is still not enough (12 teeth)
- Increasing value of k by 3 (k=162) gives satisfactory results:

$$k = 6N_2 = 162 \Rightarrow N_2 = 27$$

$$k = 9N_4 = 162 \Rightarrow N_4 = 18$$

*and*

$$N_3 = 5N_2 = 135$$

$$N_5 = 8N_2 = 144$$

# Design problem 2

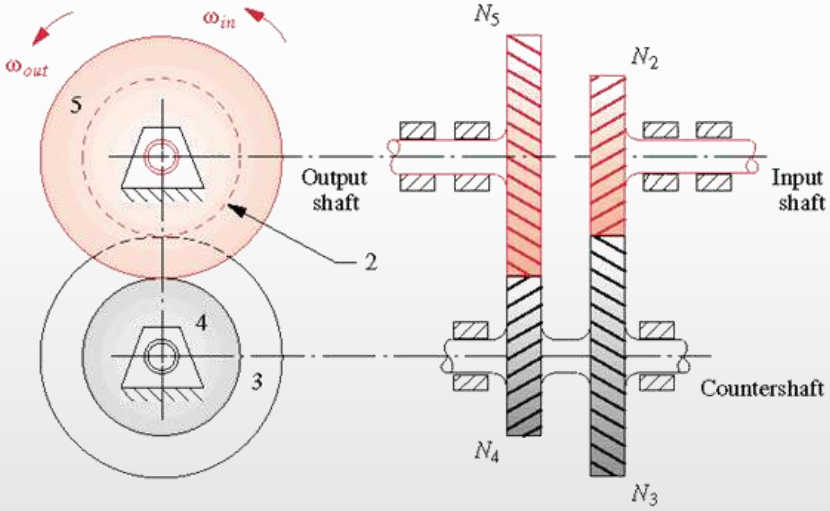
- So the final solution to the problem will be to use the following compounded gear sets:

$$N_2 = 27$$

$$N_3 = 135$$

$$N_5 = 18$$

$$N_4 = 144$$




- Which will result in

$$m_v = \frac{\omega_{out}}{\omega_{in}} = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right) = \left(-\frac{27}{135}\right)\left(-\frac{18}{144}\right) = \frac{1}{40}$$

# Design problem 2

- The pitch diameters of the gears are:

$$d_2 = \frac{N_2}{\rho_d} = 3.375\,in \qquad d_3 = \frac{N_3}{\rho_d} = 16.875\,in$$

$$d_4 = \frac{N_4}{\rho_d} = 2.25\,in \qquad d_5 = \frac{N_5}{\rho_d} = 18\,in$$

- Which will result in a center distance of 10.125"